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Note

Dense ellipsoid packings

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Abstract

The densest lattice packings of ellipsoids in Euclidian d-space $\mathbb{E}^d$ are known for $d \leqslant 8$. We construct dense non-lattice packings for $4 \leqslant d \leqslant 8$ that exceed the densities of the densest lattice packings by surprisingly large factors, in particular in $\mathbb{E}^8$ by more than 42.9%.

Keywords: Lattice packings; Non-lattice packings; Ellipsoids

1. Introduction

A family $\mathscr{P}$ of congruent ellipsoids in $\mathbb{E}^d$ is called a *packing* if the members of $\mathscr{P}$ have mutually disjoint interiors. The density of $\mathscr{P}$ is, intuitively speaking, the ratio of the volume covered by $\mathscr{P}$ in an arbitrarily large region and the volume of this region. If we restrict ourselves to packings with translates of an ellipsoid, we may as well consider packings of spheres since the density is affinely invariant. This is especially true if we restrict ourselves to *lattice packings*, packings of an ellipsoid translated by all vectors of a lattice.

Keplers famous conjecture states that in Euclidean 3-space $\mathbb{E}^3$ there is no non-lattice packing of spheres denser than the densest lattice packing. While Keplers problem appears to be solved (cf. [4,9]), the analogous question in $\mathbb{E}^d$, $d \geqslant 4$, is still open. In other words, if δ_d^{L} denotes the *density of the densest lattice packing of spheres* (or equivalently: of ellipsoids) in $\mathbb{E}^d$, and δ_d^{T} the *density of the densest packing of translates of spheres* (or ellipsoids), then $\delta_d^{\mathrm{T}} \geqslant \delta_d^{\mathrm{L}}$, and for $d \geqslant 4$ it is not known if equality holds or not.

Now, ellipsoids have a much larger variety of dense packings than spheres. Let δ_d^{C} denote the *supremum of packing densities of congruent ellipsoids*. Then a simple and

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elegant method by Bezdek and Kuperberg [1] shows that $\delta_d^{\mathrm{C}} > \delta_d^{\mathrm{L}}$ for all $d \geqslant 3$. For $d=2$ we have $\delta_d^{\mathrm{C}} = \delta_d^{\mathrm{T}} = \delta_d^{\mathrm{L}}$ (cf. [3]). Note that δ_d^{L} and δ_d^{T} are independent of the shape of the ellipsoids, whereas δ_d^{C} denotes the supremum of packing densities with respect to all possible shapes of ellipsoids.

Because of the difficulties in the proof of Keplers conjecture, it is clear that the exact values for δ_d^{C} might be very hard to obtain. But, since δ_d^{L} is known for $d \leqslant 8$ (cf. [9]), it is possible to get good lower bounds for δ_d^{C}, $d \leqslant 8$. For $d=3$ Bezdek and Kuperberg showed $\delta_3^{\mathrm{C}} \geqslant 0.7533\ldots$. Wills [10] improved this to $0.7585\ldots$. Still, no bounds are known for $d \geqslant 4$. In this paper we give constructions of dense ellipsoid packings in $\mathbb{E}^d$, $4 \leqslant d \leqslant 8$, which lead to the following theorem.

Theorem 1.

$$\delta_4^{\mathrm{C}} \geqslant 0.6541\ldots, \quad \delta_5^{\mathrm{C}} \geqslant 0.5531\ldots, \quad \delta_6^{\mathrm{C}} \geqslant 0.4779\ldots,$$

$$\delta_7^{\mathrm{C}} \geqslant 0.4075\ldots \quad \textit{and} \quad \delta_8^{\mathrm{C}} \geqslant 0.3625\ldots .$$

If one compares these bounds for δ_d^{C} with the known values δ_d^{L}, then one gets a lower bound for the *relative difference* $\Delta_d = (\delta_d^{\mathrm{C}} - \delta_d^{\mathrm{L}})/\delta_d^{\mathrm{L}}$. These and the other values are listed in Section 3. Here, in particular $\Delta_8 \geqslant 0.4291\ldots$ means that in $\mathbb{E}^8$ there are packings with congruent ellipsoids which are more than 42.9% denser than the densest lattice packing, which is comparatively dense itself. The proof of the theorem is given in Sections 2 and 3.

2. Constructions from a given packing lattice

Throughout this paper, let B_r^d denote the d-dimensional sphere of radius r. In particular $B^d = B_1^d$ is the unit sphere. Let $\Lambda \subset \mathbb{E}^d$ be a *packing lattice* (cf. [2]) for B_r^d. Then, the *density* of the lattice packing $\Lambda + B_d(r)$ is given by $\delta(\Lambda + B_r^d) = V_d(B_r^d)/\det(\Lambda)$. Here, $V_d(\cdot)$ denotes the d-dimensional *volume* and $\det(\cdot)$ the *determinant* of a lattice. Horváth [5] showed that any lattice packing of spheres (ellipsoids) in $\mathbb{E}^d$, $d \geqslant 3$, leaves *tunnels* (infinite cylinders) of free space. The idea of Bezdek and Kuperberg was to fill these tunnels with translates of an ellipsoid of volume $V_d(B_r^d)$, and then apply the following Lemma to construct a denser non-lattice packing of congruent ellipsoids.

Lemma 2 (Bezdek and Kuperberg [1]). *Given two ellipsoids $E_1, E_2 \in \mathbb{E}^d$ with $V_d(E_1) = V_d(E_2)$ there exists an affinity $A : \mathbb{E}^d \to \mathbb{E}^d$ such that $A(E_1)$ and $A(E_2)$ are congruent.*

Here, we consider tunnels in both lattice and non-lattice packings, generalizing the constructions of Bezdek, Kuperberg and Wills. This leads to the conjecture that the construction from a non-lattice packing yields a better ellipsoid packing only for $d=3$. Remarks on this are in Section 4.

Given a lattice packing $\Lambda + B_r^d$, the orthogonal projection $(\Lambda + B_r^d \mid H)$ onto a hyperplane H may have a gap $H \backslash (\Lambda + B_r^d \mid H)$. This gap results from tunnels of uncovered space. If we fill these tunnels with arbitrarily thin ellipsoids, we can get a density in the tunnels arbitrarily close to δ_d^{L} (or δ_d^{T}). Then, the density of the constructed ellipsoid packings becomes arbitrarily close to

$$\delta_{\Lambda,H} = \delta(\Lambda + B_r^d) + (1 - \delta_H)\delta_d^{\mathrm{L}}, \tag{1}$$

where δ_H denotes the density of the projection onto H. Hence, we want to find a hyperplane H onto which the projection has small density.

For a given lattice Λ, we consider a *lattice hyperplane* H, a hyperplane containing a sublattice Λ_{s} which is not contained in any subspace of lower dimension. A sublattice Λ_{s} is called *dense*, if there exists a subspace S of $\mathbb{E}^d$ with $\Lambda \cap S = \Lambda_{\mathrm{s}}$. Now, given the dense sublattice $\Lambda_{\mathrm{s}} = \Lambda \cap H$, the set $\mathscr{L} = \Lambda_{\mathrm{s}} + B_r^d$ is called a *layer* of $\Lambda + B_r^d$. Note that every lattice packing is a *stacking* of layers $\bigcup_{n\in\mathbb{Z}}(nv + \mathscr{L})$ ($\mathbb{Z}$ is the set of integers), with a suitable $v \in \Lambda$. In some cases it is possible to rearrange the layers to form a non-lattice packing $\bigcup_{i\in\mathbb{Z}}(v_i + \mathscr{L})$, $v_i \in \mathbb{E}^n$, with the same or even better density. If the density of the projection onto H is smaller for the latter packing, we can construct ellipsoid packings with even higher density from it.

Given a layer $\mathscr{L}$ and corresponding lattice hyperplane H, we stack a copy $v + \mathscr{L}$ adjacent as in the lattice packing $\Lambda + B_r^d$. Now, let $w = p_v - v$ be the difference between v and its projection $p_v = (v \,|\, H)$ onto H. Then the set

$$\mathscr{P} = \bigcup_{n\in\mathbb{Z}} (2nw + ((v + \mathscr{L}) \cup \mathscr{L}))$$

is a packing for B_r^d with density $\delta(\Lambda + B_r^d)$, if the length $\|w\|$ of w satisfies $\|w\| \geqslant r$. Otherwise the spheres of two layers, adjacent to another layer may overlap. Because of $\|w\| = \det(\Lambda)/\det(\Lambda_{\mathrm{s}})$, the condition is equivalent to

$$\det(\Lambda_{\mathrm{s}}) \leqslant \frac{\det(\Lambda)}{r}. \tag{2}$$

To find sublattices $\Lambda_{\mathrm{s}} \subset \Lambda$ with (2) we use the *adjoint lattice*

$$\Lambda^* = \det(\Lambda)\{x \in \mathbb{E}^d \mid \langle x, y\rangle \in \mathbb{Z}, y \in \Lambda\}$$

of Λ. Here, $\langle\cdot,\cdot\rangle$ denotes the inner product of $\mathbb{E}^d$. We use Λ^* because every dense k-dimensional sublattice of Λ corresponds to an orthogonal dense $(d-k)$-dimensional sublattice of Λ^* with the same determinant (cf. [8]). In particular, a dense sublattice $\Lambda_{\mathrm{s}} = \Lambda \cap H$ in a lattice hyperplane H of Λ corresponds to a *primitive lattice point* (generator for a 1-dimensional sublattice) of Λ^* with length $\det(\Lambda_{\mathrm{s}})$.

Given a sublattice $\Lambda_{\mathrm{s}} = \Lambda \cap H$ with (2), $\mathscr{P}$ is the lattice packing $\Lambda + B_r^d$ or a non-lattice packing with the same density. In both the cases, the projection onto H is given by $(\Lambda_{\mathrm{s}} + B_r^{d-1}) \cup (p_v + \Lambda_{\mathrm{s}} + B_r^{d-1})$. If $x_1, \ldots, x_k \in \Lambda_{\mathrm{s}}$ are all the points of Λ_{s} with $h_i = r - \|x_i - p_v\|/2 > 0$, then the projection has the density

$$\delta_H = \frac{2V_{d-1}(B_r^{d-1}) - 2\sum_{i=1}^{k} V_{d-1}(C_r^{d-1}(h_i))}{\det(\Lambda_{\mathrm{s}})} \tag{3}$$

since any point in the projection is in the interior of at most two spheres. Here, $C_r^d(h)$ denotes a sphere-cap of height h, taken from B_r^d. With κ_d being the volume of the unit sphere in $\mathbb{E}^d$, we have

$$V_d(C_r^d(h)) = \int_{r-h}^{r} V_{d-1}\left(B^{d-1}_{\sqrt{r^2-x^2}}\right)\,\mathrm{d}x = \kappa_{d-1}\int_{r-h}^{r}(r^2-x^2)^{(d-1)/2}\,\mathrm{d}x.$$

3. Constructions from densest lattice packings

For a given lattice packing $\Lambda + B_r^d$ we may now evaluate the densities $\delta_{\Lambda,\Lambda_s} = \delta_{\Lambda,H}$ for all sublattices $\Lambda_s = \Lambda \cap H$ satisfying (2). These values give a lower bound for δ_d^{C}. Hence, verification of the values in Table 1 proves the theorem. For our constructions we use the densest packing lattices, which are known for $d \leqslant 8$ only. For detailed descriptions of the occurring lattices we refer to Table 1 [2].

The densest lattice packings of spheres in Euclidian 3-, 4-, and 5-space are attained by the *checkerboard lattice* D_d. For spheres of radius $r_0 = \sqrt{2}/2$, generator matrices for D_d and its adjoint D_d^* are given by

$$M(D_d) = \begin{pmatrix} -1 & 0 & \cdots & 0 & 1 \\ 0 & -1 & \cdots & 0 & 1 \\ \cdots & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & -1 & 1 \\ 0 & 0 & \cdots & 0 & 2 \end{pmatrix}, \quad M(D_d^*) = \begin{pmatrix} -2 & 0 & \cdots & 0 & 0 \\ 0 & -2 & \cdots & 0 & 0 \\ \cdots & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & -2 & 0 \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix}.$$

Because of (2) we only need to consider all primitive points of D_d^* with length $\leqslant 2\sqrt{2}$. Up to a change of columns and/or signs there are only three (two for $d > 8$) different possibilities, namely $v_1^* = (2,0,\ldots,0)$, $v_2^* = (2,2,0,\ldots,0)$ and $v_3^* = (1,\ldots,1)$.

For $d = 3$, the choice of v_2^* leads to the construction by Bezdek and Kuperberg, the choice of v_3^* to the one by Wills. For the latter construction we may take the first two rows of $M(D_3)$ to span our sublattice Λ_s, which is the known *hexagonal lattice* A_2. $\mathscr{P}$ is a uniform non-lattice packing in $\mathbb{E}^3$. The projection points $p_v + \Lambda_s$ (e.g., v equal to the third row of $M(D_3)$) onto the plane H containing A_2 have the distance $\sqrt{6}/3$ to three points of A_2. With (1) and (3) we find $\delta_{D_3,A_2} = 0.7585\ldots$.

Table 1

d	δ_d^{L}	Λ	Λ_s	$\delta_{\Lambda,\Lambda_s}$	$\Delta_d \geqslant$
3	0.7404…	D_3	A_2	0.7585…	0.0243…
4	0.6168…	D_4	$\mathbb{Z}^3$	0.6541…	0.0605…
5	0.4652…	D_5	S_4	0.5531…	0.1890…
6	0.3729…	E_6	A_5	0.4779…	0.2814…
7	0.2952…	E_7	D_6	0.4075…	0.3802…
8	0.2536…	E_8	E_7	0.3625…	0.4291…

For $d=4$ and 5 the choice of v_2^*, and hence, the sublattices with determinant $2\sqrt{2}$ turn out to be best possible. We may take, e.g., the sublattice spanned by the difference of the first two rows together with the last two, respectively, three rows of $M(D_d)$. For $d=4$ this sublattice is a copy of $\sqrt{2}\mathbb{Z}^3$. The projection points lie in the centers of cubes $[0,\sqrt{2}]^3$, having a distance of $\sqrt{6}/2$ to the eight vertices. For $d=5$ the sublattice, say S_4, consists of stacked copies of $\mathbb{Z}^3$. Here, S_4 may as well be viewed as an orthogonal stacking with copies of D_3 $(=A_3)$. The projection points lie between two *deep holes* (cf. [2]) of D_3-layers, having a distance of $\sqrt{6}/2$ to 12 points of S_4.

To give an example, we verify the value of δ_{D_5,S_4}. By (1) we have $\delta_{D_5,S_4}=(2-\delta_H)\delta_5^{\mathrm{L}}$. Because of (3) we need to evaluate $V_4(C_{r_0}^4(h_i))$ for $h_i=r_0-\sqrt{6}/4$:

$$V_4(C_{r_0}^4(h_i))=\kappa_3\int_{\sqrt{6}/4}^{\sqrt{2}/2}\left(\frac{1}{2}-x^2\right)^{3/2}\mathrm{d}x=\frac{4\pi}{3}\int_{\pi/3}^{\pi/2}\frac{1}{4}\cos^4(y)\,\mathrm{d}y=\frac{4\pi^2-7\sqrt{3}\pi}{192}.$$

Now δ_H is given by

$$\delta_H=\frac{2r_0^4\kappa_4-24V_4(C_{r_0}^4(h_i))}{2\sqrt{2}}=\frac{7\sqrt{6}\pi-2\pi^2}{32}=0.8109\ldots\ ,$$

which yields the asserted value of δ_{D_5,S_4}.

The choice of v_3^* for $d=5$ leads to the construction of a non-lattice packing. Here, in contrast to the case $d=3$, the gap in the projection is smaller than the one calculated above. Calculations show that the same is true for projections of uniform non-lattice packings found by Leech [6].

Now, we turn to dimensions 6–8. The densest lattice packing of spheres (with radius r_0) in $\mathbb{E}^8$ is attained by the lattice $E_8=D_8\cup(\frac{1}{2},\ldots,\frac{1}{2})+D_8$ with generator matrix

$$M(E_8)=M(E_8^*)=\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0\\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}.$$

The first equality holds because of $E_8^*=E_8$. The densest lattice packings of spheres in $\mathbb{E}^6$ and $\mathbb{E}^7$ are attained by sublattices E_6 and E_7 of E_8, generated by the latter six, and respectively seven rows of $M(E_8)$.

For $d=8$, we have to choose a vector of E_8 with length $\leqslant\sqrt{2}$, because of (2). Any vector of $E_8^*=E_8$ has at least this length, and there is only one lattice perpendicular to these, namely E_7 (cf. [2]). The projection points have a distance of $\sqrt{6}/2$ to 56 points of E_7. Again, with (1) and (3) we verify the values in the table.

For $d=6$ and 7 we find sublattices Λ_s satisfying (2) by choosing between those perpendicular to 3-dimensional and respectively, 2-dimensional sublattices of E_8 which have an appropriate determinant. For $d=6$ we have to consider all 3-dimensional sublattices of E_8 with determinant $\leqslant \sqrt{6}$ because the determinant of E_6 is $\sqrt{3}$. There are two possibilities: one is the lattice D_3, the other is a lattice obtained by an orthogonal stacking with copies of the hexagonal lattice A_2. The first case leads to the construction of a uniform non-lattice packing with layers of D_5. This packing is described in [7] by Leech. In the second case, we use a sublattice known as A_5 in our construction, leading to a better density. A possible base is given by the third to the seventh row of $M(E_8)$. The projection points have a distance of $\sqrt{6}/2$ to 20 points of A_5. Note that, again, the projection of the non-lattice packing is denser than that of the lattice packing given by E_6.

For $d=7$, we consider all 2-dimensional sublattices of E_8 with determinant $\leqslant 2$. Again, we have two choices: one is the hexagonal lattice A_2, the other is the lattice $\sqrt{2}\mathbb{Z}^2$. The first choice leads to the construction of a uniform non-lattice packing, which is described in [7] as well. The second choice for Λ_s is the lattice D_6, generated, e.g., by the second to the fifth, together with the seventh row of $M(E_8)$. The projection points have a distance of $\sqrt{6}/2$ to 32 points of D_6. Here, as before, the use of a lattice packing in the construction yields a denser packing.

4. Remarks

The values for $d=4,\dots,8$, listed in Section 3, can be obtained, as well, by projecting the densest lattice packings in suitable directions because the constructions from lattice packings turned out to be denser for these d. In this way we may regard $\mathbb{E}^3$ as exceptional because Wills construction is based on a non-lattice packing.

Instead of δ_d^{C}, we may define $\delta_d^{\mathrm{C},i}$ as the *supremum of densities of packings with i translation types of congruent ellipsoids*. Then we have $\delta_d^{\mathrm{C}} = \delta_d^{\mathrm{C},\infty} \geqslant \cdots \geqslant \delta_d^{\mathrm{C},i+1} \geqslant \delta_d^{\mathrm{C},i} \geqslant \cdots \geqslant \delta_d^{\mathrm{C},1} = \delta_d^{\mathrm{T}}$, and, the bounds in Theorem 1 hold for $\delta_d^{\mathrm{C},2}$. We conjecture that these bounds are either tight or not far from the best values. It is unknown if $\delta_d^{\mathrm{C},i}$ differs for $i \geqslant 2$. Especially, the question if $\delta_3^{\mathrm{C}} > \delta_3^{\mathrm{C},2}$ holds seems to be interesting.

We do not know non-trivial upper bounds for δ_d^{C}, but we have $\delta_d^{\mathrm{C},i+j} \leqslant \delta_d^{\mathrm{C},i} + \delta_d^{\mathrm{C},j}$. In particular $\delta_d^{\mathrm{C},2} \leqslant 2\delta_d^{\mathrm{T}}$ shows that packings with two types of congruent ellipsoids, as constructed in this paper, may be at most 100% denser than the densest packing with translates. It is open whether or not there exists a dimension d with $\delta_d^{\mathrm{C},2}$ close to $2\delta_d^{\mathrm{T}}$.

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